EXERCISES

1. (For the ϵ-Neighborhood Theorem) Show that any neighborhood $\tilde{U}$ of Y in $\mathbf{R}^M$ contains some Y^ϵ; moreover, if Y is compact, ϵ may be taken constant. [HINT: Find covering open sets $U_\alpha \subset Y$ and $\epsilon_\alpha > 0$, such that $U_\alpha^{\epsilon_\alpha} \subset \tilde{U}$. Let $\{\theta_i\}$ be a subordinate partition of unity, and show that $\epsilon = \sum \theta_i \epsilon_i$ works.]

2. Let Y be a compact submanifold of $\mathbf{R}^M$, and let $w \in \mathbf{R}^M$. Show that there exists a (not necessarily unique) point $y \in Y$ closest to w, and prove that $w - y \in N_y(Y)$. [HINT: If $c(t)$ is a curve on Y with $c(0) = y$, then the smooth function $|w - c(t)|^2$ has a minimum at 0. Use Exercise 12, Chapter 1, Section 2.]

3. Use Exercise 2 to verify the geometric characterization of $\pi: Y^\epsilon \longrightarrow Y$, for compact Y. Assume that $h: N(Y) \longrightarrow \mathbf{R}^M$ carries a neighborhood of Y in $N(Y)$ diffeomorphically onto Y^ϵ, ϵ constant. Prove that if $w \in Y^\epsilon$, then $\pi(w)$ is the *unique* point of Y closest to w.

4. (General Position Lemma) Let X and Y be submanifolds of $\mathbf{R}^N$. Show that for almost every $a \in \mathbf{R}^N$ the translate $X + a$ intersects Y transversally.

***5.** Suppose that the compact submanifold X in Y intersects another submanifold Z, but $\dim X + \dim Z < \dim Y$. Prove that X may be pulled away from Z by an arbitrarily small deformation: given $\epsilon > 0$ there exists a deformation $X_t = i_t(X)$ such that X_1 does not intersect Z and $|x - i_1(x)| < \epsilon$ for all $x \in X$. (*Note*: You need Exercise 11, Chapter 1, Section 6. The point here is to make X_t a manifold.)

6. Sharpen Exercise 5. Assume that Z is closed in Y and let U be any open set in X containing $Z \cap X$. Show that the deformation X_t may be chosen to be constant outside of U (Figure 2-11).

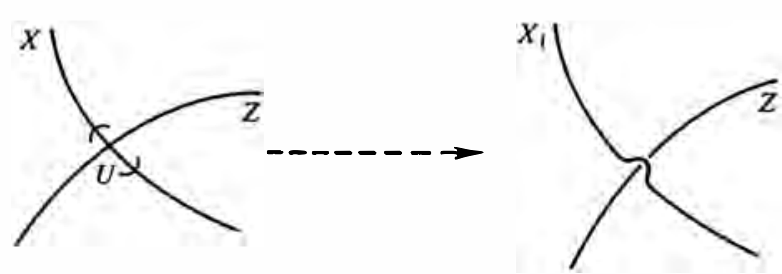

Figure 2-11

*7. Suppose that X is a submanifold of $\mathbf{R}^N$. Show that "almost every" vector space V of any fixed dimension l in $\mathbf{R}^N$ intersects X transversally. [HINT: The set $S \subset (\mathbf{R}^N)^l$ consisting of all linearly independent l-tuples of vectors in $\mathbf{R}^N$ is open in $\mathbf{R}^{Nl}$, and the map $\mathbf{R}^l \times S \longrightarrow \mathbf{R}^N$ defined by

$$[(t_1, \ldots, t_l), v_1, \cdots, v_l] \longrightarrow t_1 v_1 + \ldots + t_l v_l$$

is a submersion.]

8. Suppose that $f\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is a smooth map, $n > 1$, and let $K \subset \mathbf{R}^n$ be compact and $\epsilon > 0$. Show that there exists a map $f'\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ such that df'_x is never zero, but $|f - f'| < \epsilon$ on K. Prove that this result is false for $n = 1$. [HINT: Let $M(n) = \{n \times n \text{ matrices}\}$, and show that the map $F\colon \mathbf{R}^n \times M(n) \longrightarrow M(n)$, defined by $F(x, A) = df_x + A$, is a submersion. Pick A so that $F_A \pitchfork \{0\}$; where is $n > 1$ used?]

9. Let $f\colon \mathbf{R}^k \longrightarrow \mathbf{R}^k$, and, for each $a \in \mathbf{R}^k$, define

$$f_a(x) = f(x) + a_1 x_1 + \cdots + a_k x_k.$$

Prove that for almost all $a \in \mathbf{R}^k$, f_a is a Morse function. [HINT: Consider

$$\mathbf{R}^k \times \mathbf{R}^k \longrightarrow \mathbf{R}^k, \qquad (x, a) \longrightarrow \frac{\partial f}{\partial x_i} + a_1, \ldots, \frac{\partial f}{\partial x_k} + a_k.$$

Show that this is a submersion, hence $\pitchfork \{0\}$.]

10. Let X be an $n - 1$ dimensional submanifold of $\mathbf{R}^n$, a "hypersurface." A point in $\mathbf{R}^n$ is called a *focal point* of X if it is a critical value of the normal bundle map $h\colon N(X) \longrightarrow \mathbf{R}^n$, $h(x, v) = x + v$. Locate the focal points of the parabola $y = x^2$ in $\mathbf{R}^2$. [*Answer:* You get a curve with a cusp at $(0, \frac{1}{2})$.]

11. Let X be a one-dimensional submanifold of $\mathbf{R}^2$, and let $p \in X$. Choose linear coordinates in $\mathbf{R}^2$ so that p is the origin, the x axis is the tangent line to X at p, and the y axis is the normal line. Show that in a neighborhood of $p = 0$, X is the graph of a function $y = f(x)$ with $f(0) = 0$ and $f'(0) = 0$. The quantity $f''(0)$ is called the *curvature* of X at p, denoted $\kappa(p)$. Show that if $\kappa(p) \neq 0$ then X has a focal point along the normal line at distance $1/\kappa(p)$ from p. [HINT: Show that the normal space to X at a point x near p is spanned by $(-f'(x), 1)$. Now compute the normal bundle map $h\colon N(X) \longrightarrow \mathbf{R}^2$.]

*12. Let Z be a submanifold of Y, where $Y \subset \mathbf{R}^M$. Define the *normal bundle to Z in Y* to be the set $N(Z; Y) = \{(z, v) : z \in Z, v \in T_z(Y) \text{ and } v \perp T_z(Z)\}$. Prove that $N(Z; Y)$ is a manifold with the same dimension as Y. [HINT: Let $g_1, \ldots, g_l$ be independent functions in a neighborhood $\tilde{U}$ of z in $\mathbf{R}^M$, with

$$U = Z \cap \tilde{U} = \{g_1 = 0, \ldots, g_l = 0\}$$

and

$$Y \cap \tilde{U} = \{g_{k+1} = 0, \ldots, g_l = 0\}$$

(Exercise 4, Chapter 1, Section 4). Show that the associated parametrization $U \times \mathbf{R}^l \longrightarrow N(Z; \mathbf{R}^M)$ as constructed in the text restricts to a parametrization $U \times \mathbf{R}^k \longrightarrow N(Z; Y)$.]

13. Consider S^{k-1} as a submanifold of S^k via the usual embedding mapping $(x_1, \ldots, x_k) \longrightarrow (x_1, \ldots, x_k, 0)$. Show that at $p \in S^{k-1}$ the orthogonal complement to $T_p(S^{k-1})$ in $T_p(S^k)$ is spanned by the vector $(0, \ldots, 0, 1)$. Prove that $N(S^{k-1}; S^k)$ is diffeomorphic to $S^{k-1} \times \mathbf{R}$.

14. Prove that the map $\sigma : N(Z; Y) \longrightarrow Z$, $\sigma(z, v) = z$, is a submersion. What specifically is the preimage $\sigma^{-1}(z)$, which we denote by $N_z(Z; Y)$?

15. Show that the map $z \longrightarrow (z, 0)$ embeds Z as a submanifold of $N(Z; Y)$.

*16. *Tubular Neighborhood Theorem.* Prove that there exists a diffeomorphism from an open neighborhood of Z in $N(Z; Y)$ onto an open neighborhood of Z in Y. [HINT: Let $Y^\epsilon \xrightarrow{\pi} Y$ be as in the ϵ-Neighborhood Theorem. Consider the map $h: N(Z; Y) \longrightarrow \mathbf{R}^M$, $h(z, v) = z + v$. Then $W = h^{-1}(Y^\epsilon)$ is an open neighborhood of Z in $N(Z; Y)$. The sequence of maps $W \xrightarrow{h} Y^\epsilon \xrightarrow{\pi} Y$ is the identity on Z, so use Exercise 14 of Chapter 1, Section 8.]

17. Let Δ be the diagonal in $X \times X$. Show that the orthogonal complement to $T_{(x,x)}(\Delta)$ in $T_{(x,x)}(X \times X)$ is the collection of vectors $\{(v, -v) : v \in T_x(X)\}$. [See Exercise 10, Chapter 1, Section 2.]

*18. Prove that the map $T(X) \longrightarrow N(\Delta; X \times X)$, defined by sending (x, v) to $((x, x), (v, -v))$, is a diffeomorphism. Use the Tubular Neighborhood Theorem to conclude that there is a diffeomorphism of a neighborhood of X in $T(X)$ with a neighborhood of Δ in $X \times X$, extending the usual diffeomorphism $X \longrightarrow \Delta$, $x \longrightarrow (x, x)$.

*19. Let Z be a submanifold of codimension k in Y. We say that the normal bundle $N(Z; Y)$ is *trivial* if there exists a diffeomorphism $\Phi : N(Z; Y)$

$\longrightarrow Z \times \mathbf{R}^k$ that restricts to a linear isomorphism $N_z(Z; Y) \longrightarrow \{z\} \times \mathbf{R}^k$ for each point $z \in Z$. As a check on your grasp of the construction, prove that $N(Z; Y)$ is always *locally trivial*. That is, each point $z \in Z$ has a neighborhood V in Z such that $N(V; Y)$ is trivial.

***20.** Prove that $N(Z; Y)$ is trivial if and only if there exists a set of k independent global defining functions $g_1, \ldots, g_k$ for Z on some set U in Y. That is,

$$Z = \{y \in U : g_1(y) = 0, \ldots, g_k(y) = 0\}.$$

[HINT: If $N(Z; Y)$ is trivial, then there obviously exist global defining functions for Z in $N(Z; Y)$. Transfer these functions to an open set in Y via the Tubular Neighborhood Theorem, Exercise 16. Conversely, if there exists a submersion $g : U \longrightarrow \mathbf{R}^k$ with $g^{-1}(0) = Z$, check that, for each $z \in Z$, the transpose map $dg_z^t : \mathbf{R}^k \longrightarrow T_z(Y)$ carries $\mathbf{R}^k$ isomorphically onto the orthogonal complement of $T_z(Z)$ in $T_z(Y)$; thus $\Phi^{-1} : Z \times \mathbf{R}^k \longrightarrow N(Z; Y)$ is defined by $\Phi^{-1}(z, a) = (z, dg_z^t a)$.]

§4 Intersection Theory Mod 2

The previous section was technical and rather difficult. We now hope to convince you that the effort was worth it. In this section we will use the transversality lemma and the other results of Section 3 to develop a simple intuitive invariant for intersecting manifolds, from which we will be able to obtain many nice geometric consequences.

Two submanifolds X and Z inside Y have *complementary dimension* if $\dim X + \dim Z = \dim Y$. If $X \pitchfork Z$, this dimension condition makes their intersection $X \cap Z$ a zero-dimensional manifold. (We are working now without boundaries.) If we further assume that both X and Z are closed and that at least one of them, say X, is compact, then $X \cap Z$ must be a finite set of points. Provisionally, we might refer to the number of points in $X \cap Z$ as the "intersection number" of X and Z, indicated by $\#(X \cap Z)$. See Figure 2-12.

How can we generalize this discussion to define the intersection number of the compact X with an arbitrary closed Z of complementary dimension?

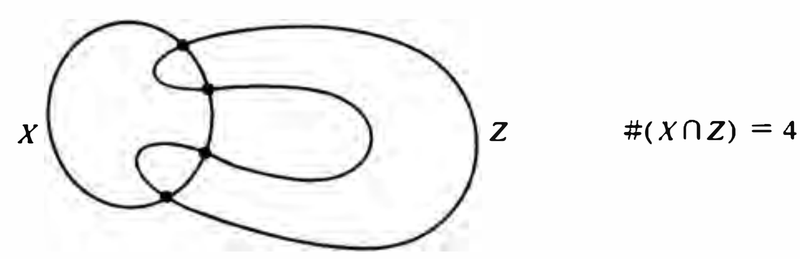

Figure 2-12